ON TUNNEL NUMBER ONE KNOTS THAT ARE NOT $(1, n)$

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Abstract. We show that the bridge number of a t bridge knot in S^3 with respect to an unknotted genus t surface is bounded below by a function of the distance of the Heegaard splitting induced by the t bridges. It follows that for any natural number n, there is a tunnel number one knot in S^3 that is not $(1, n)$.

1. Introduction

A compact, connected, closed, orientable surface S embedded in S^3 is *standardly embedded* if the closure of each component of its complement is a handlebody. Equivalently, S is a Heegaard surface for S^3. A knot K is in n-bridge position with respect to S if the intersection of K with each handlebody is a collection of n boundary parallel arcs.

For $n \geq 1$, we will say that K is (t, n) if K can be put in n-bridge position with respect to a standardly embedded, genus t surface S. We will say that K is $(t, 0)$ if K can be isotoped into S. If K is (t, n) for some n then K is (t, m) for every $m \geq n$. Thus the important number is the smallest n such that K is (t, n).

A set of arcs properly embedded in the the complement of a knot K is an *unknotting system* if the complement of a regular neighborhood of K and the arcs is a handlebody. The *tunnel number* of K is the minimum number of arcs in an unknotting system for K.

Let K be a knot in S^3 and Σ the Heegaard splitting of the knot complement induced by a t-tunnel decomposition for K. Hempel defined a distance $d(\Sigma)$ for Heegaard splittings using the curve complex. We will prove the following:

1. **Theorem.** *If K is (t, n) then K is $(t, 0)$ or $d(\Sigma) \leq 2n + 2t$.*

Every tunnel number t knot is $(t + 1, 0)$. The question is for what values of n can a tunnel number t knot be (t, n). Moriah and Rubinstein [7] showed that there exist tunnel number one knots that are

1991 *Mathematics Subject Classification.* Primary 57M.
Key words and phrases. Tunnel number one, bridge number, curve complex.
This research was supported by NSF VIGRE grant 0135345.

$(1, 2)$, but not $(1, 1)$. Morimoto, Sakuma and Yokota [8] and Eudave-Muñoz [1] constructed further examples of knots that are not $(1, 1)$. Eudave-Muñoz has recently announced the existence of tunnel number one knots that are not $(1, 2)$. The first author of this paper [4] showed that for tunnel number one knots, $d(\Sigma)$ can be arbitrarily large. Thus Theorem 1 implies the following:

2. **Corollary.** *For every $n \in \mathbf{N}$, there is a tunnel number one knot K such that K is not $(1, n)$.*

The proof in [4] is non-constructive and therefore does not provide actual examples of knots with high toroidal bridge number. Since this note first appeared as a preprint, Minsky, Moriah and Schleimer [6] have given a constructive proof that there are t-tunnel knots in S^3 with arbitrarily high distance splittings. They conclude, using Theorem 1, that for every t and k, there is a t tunnel knot that is not (t, k).

We describe weakly incompressible surfaces in Section 2 and the curve complex in Section 3. Theorem 1 and Corollary 2 are proved in Section 4.

2. Weakly Compressible Surfaces

A properly embedded, two sided surface S in a 3-manifold M is *compressible* if there is a disk D in M such that ∂D is an essential simple closed curve in S and the interior of D is disjoint from S. If S is not compressible then S is *incompressible*.

Assume that S separates M into components X and Y. Then S is *strongly compressible* if there are disks D_1 and D_2 such that ∂D_1 and ∂D_2 are disjoint, essential simple closed curves in S, the interior of D_1 is contained in X (disjoint from S) and the interior of D_2 is contained in Y. If S is not strongly compressible then S is *weakly incompressible*.

A properly embedded surface S is *boundary compressible* if there is a disk $D \subset M$ such that ∂D consists of an essential arc in S and an arc in ∂M. A separating surface S is *strongly boundary compressible* if there are boundary compressing disks on opposite sides of S with disjoint boundaries, or a boundary compressing disk and a compressing disk on opposite sides of S with disjoint boundaries. A surface is *weakly boundary incompressible* if S is not strongly boundary compressible and S is not strongly compressible.

3. **Lemma.** *Let M be a compact 3-manifold and F a closed, separating, incompressible torus embedded in M. Let A, B be the closures of the components of the complement of F. Let S be a second surface which separates M. If $S \cap A$ is weakly boundary incompressible in A and*

$S \cap B$ is empty or incompressible and boundary incompressible in B then S is weakly incompressible in M. If $S \cap A$ and $S \cap B$ are both incompressible and boundary incompressible, then S is incompressible in M.

Proof. Assume for contradiction S is strongly compressible. Then there are disks D_1, D_2 properly embedded on opposite sides of S such that $\partial D_1 \cap \partial D_2$ is empty.

Assume D_1 and D_2 have been chosen transverse to F and with a minimal number of components in $(D_1 \cup D_2) \cap F$. If D_1 and D_2 are disjoint from F then both disks must be in A because $S \cap B$ is incompressible. This contradicts the assumption that $S \cap A$ is weakly boundary incompressible. Without loss of generality, assume $F \cap D_1$ is not empty.

Because F is incompressible and any loop in D_1 is trivial in D_1, any loop component of $D_1 \cap F$ must be trivial in F. Compressing D_1 along an innermost such loop will reduce the number of components of intersection without changing its boundary. Thus minimality implies $D_1 \cap F$ is a collection of arcs. Similarly, if $D_2 \cap F$ is not empty then $D_2 \cap F$ is a collection of arcs.

An outermost arc β in D_1 cuts off a disk whose boundary consists of an arc α in F and an arc β in $S \cap A$ or $S \cap B$. If the arc β is trivial in $S \cap B$ or $S \cap A$ then it can be pushed across F (taking any other arcs with it) and reducing $(D_1 \cup D_2) \cap F$. Thus we can assume that β is essential in $S \cap A$ or $S \cap B$.

If β is in $S \cap B$ then the outermost disk is a boundary compression disk for $S \cap B$. Because $S \cap B$ is boundary incompressible, this is not possible so β must be in $S \cap A$ and D_1 contains a boundary compression disk D for $S \cap A$.

If D_2 is disjoint from F then D_2 is a compression disk for $S \cap A$. This compression disk is on the opposite side from D and ∂D is disjoint from ∂D_2. This contradicts the assumption that $S \cap A$ is weakly boundary incompressible. If D_2 intersects F then, as with D_1, an outermost disk argument implies that D_2 contains a boundary compressing disk D' for $S \cap A$. The disks D and D' are disjoint and on opposite sides of $S \cap A$, again contradicting weak boundary incompressibility.

The case in which $S \cap A$ and $S \cap B$ are both incompressible and boundary incompressible proceeds similarly, but more easily. $\square$

To apply Lemma 3 to knots, we need a result regarding thin position for a knot in the 3-sphere with respect to a standard genus g Heegaard splitting. The result follows from unpublished work of C. Feist [2]. His Theorem 5.5 implies:

4. **Lemma.** *If a knot K is (t, n) and not $(t, 0)$ then either (case 1) there is a bicompressible, weakly boundary incompressible meridinal genus t surface with at most $2n$ boundary components in the complement of K or (case 2) there is an incompressible, boundary incompressible meridinal surface with genus at most t and at most $2n$ boundary components in the complement of K.*

3. The Curve Complex

Let H be a 3-manifold with boundary and let Σ be a component of ∂H.

5. **Definition.** The *curve complex* $C(\Sigma)$ is the graph whose vertices are isotopy classes of simple closed curves in Σ and edges connect vertices corresponding to disjoint curves.

For more detailed descriptions of the curve complex, see [3] and [5].

6. **Definition.** The *boundary set* $\mathbf{H} \subset C(\Sigma)$ corresponding to H is the set of vertices $\{l \in C(\Sigma) : l \text{ bounds a disk in } H\}$.

Given vertices l_1, l_2 in $C(\Sigma)$, the distance $d(l_1, l_2)$ is the geodesic distance: the number of edges in the shortest path from l_1 to l_2. This definition extends to a definition of distances between subsets X, Y of $C(\Sigma)$ by defining $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$ and for distances between a point and a set similarly.

Given a compact, connected, orientable 3-manifold M and a compact, connected, closed, separating surface Σ, let A and B be the closures of the complement in M of Σ. Then Σ is a component of ∂A and a component of ∂B. Let X,Y be the boundary sets in $C(\Sigma)$ of A and B, respectively. If X and Y are non-empty, we will define $d(\Sigma) = d(X, Y)$.

This situation arises in a knot complement as follows: Let M be the complement of a regular neighborhood of a knot K in S^3 and let $\tau_1, \ldots, \tau_t$ be a collection of properly embedded arcs in M. The arcs $\tau_1, \ldots, \tau_t$ are called a collection of *unknotting tunnels* for K if the complement in M of a regular neighborhood N of $\bigcup \tau_i \cup \partial M$ is a handlebody. Let Σ be the boundary component of the closure of N that is disjoint from ∂M. The surface Σ separates M and allows us to define $d(\Sigma)$ as above. For $t = 1$, Lemma 4 and Lemma 11 of [4] imply the following Lemma:

7. **Lemma.** *For every N, there is a knot K in S^3 and an unknotting tunnel τ such that for Σ constructed as above $d(\Sigma) > N$.*

In [4], it is shown that $d(\Sigma)$ bounds below both the bridge number of K and the Seifert genus of K. Theorem 1 provides a similar bound for the toroidal bridge number.

4. Bounding Distance

A compact, separating surface Σ properly embedded in a manifold M is called *bicompressible* if there are compressing disks for Σ in both components of $M \setminus \Sigma$.

Given a bicompressible, weakly incompressible surface Σ, let A, B be the closures of the complements of $M \setminus \Sigma$. If we compress Σ into A, the resulting surface, Σ', separates A. It may be possible to compress Σ' still further into the component of $A \setminus \Sigma'$ which does not contain Σ, creating a new surface which again separates A.

Let Σ_A be the result of compressing Σ' away from Σ repeatedly, until the resulting surface has no compression disks on the side which does not contain Σ. Let Σ_B be the result of the same operation, but compressing Σ maximally into B. Define Σ^* to be the submanifold of M bounded by Σ_A and Σ_B. Following [9] (with slightly different notation), we will say that weakly incompressible surfaces Σ and S are *well separated* if S^* can be isotoped disjoint from Σ^*. We will say that Σ and S are parallel if S can be isotoped to be parallel to Σ. The following is Theorem 3.3 in [9].

8. **Theorem** (Scharlemann and Tomova [9])**.** *If Σ and S are bicompressible, weakly incompressible, connected, closed surfaces in M then either Σ and S are well separated, Σ and S are parallel, or $d(\Sigma) \leq 2 - \chi(S)$.*

This theorem is the key to the following proof. Note that $2 - \chi(S)$ is precisely twice the genus of S.

Proof of Theorem 1. Let M be the complement in S^3 of a neighborhood of a knot K and assume K is (t, n). By Lemma 4, there is either an incompressible, boundary incompressible or a bicompressible, weakly boundary incompressible $2k$-punctured genus t surface T properly embedded in M with $k \leq n$.

Let M' be the complement in S^3 of a neighborhood of the connect sum of k trefoil knots.

There is a collection T' of k pairwise disjoint, properly embedded, essential annuli in M' and there is a homeomorphism $\phi : \partial M \to \partial M'$ which sends ∂T onto $\partial T'$. Let M'' be the result of gluing M and M' via the map ϕ. The image in M'' of $T' \cup T$ is a closed, genus $t + k$ surface which we will call S. The Euler characteristic of S is

$2-2(k+t)$. Because T is incompressible or weakly incompressible and T' is incompressible, Lemma 3 implies that S is either incompressible or weakly incompressible.

Lemma 3 also implies that the image in M'' of Σ is weakly incompressible because Σ is weakly incompressible in M and $\Sigma \cap M'$ is empty.

Suppose $T' \cup T$ is compressible but weakly incompressible. Then by Theorem 8, either Σ and S are parallel, the surfaces are well-separated or $d(\Sigma) \leq 2(k+t) \leq 2n+2t$. To complete the proof of this case we will show that Σ and S are not parallel or well separated.

First we will show that the surfaces are not parallel. The surface Σ bounds a submanifold containing the closed, incompressible torus ∂M. If Σ and S are parallel then the complement of S contains an incompressible torus A, isotopic to ∂M. Assume for contradiction this is the case. Any loop in the intersection $A \cap \partial M$ must be trivial in both surfaces or essential in both, as both surfaces are incompressible. Any trivial loop of intersection can be eliminated by an isotopy of A which keeps A disjoint from S, so we can assume $A \cap S$ is empty or consists of essential loops.

If $A \cap S$ is empty then A is contained in M or M'. If M contains an essential torus then as noted in [10], $d(\Sigma) \leq 2$ and we are done. Thus we will assume the only incompressible surface in M is boundary parallel. Such a surface cannot be disjoint from $T \subset S$.

Each component of the complement in M' of T' is homeomorphic to an unknot complement or a trefoil knot complement. Thus an incompressible surface in M' which does not intersect T' bounds an unknot complement or a trefoil complement. If ∂M is isotopic to one of these surfaces, then M must be an unknot or trefoil complement. In either case, $d(\Sigma) \leq 2$ (see [4]). Thus we will assume $A \cap S$ must be non-empty.

Let A' be a component of $A \cap M$. An incompressible annulus properly embedded in M is always boundary parallel, so one component of $M \setminus A'$ is a solid torus. The surface S cannot be contained in this solid torus, so A' can be isotoped across ∂M, reducing $A \cap \partial M$. This implies A is disjoint from ∂M, which we saw above is a contradiction. Hence A and Σ are not parallel.

To show that the surfaces are not well separated, consider the subsets Σ^* and S^* of M'' defined above. The surface Σ compresses down to a ball on one side and to a neighborhood of ∂M on the other, so we can take Σ^* to be the image in M'' of M. If Σ and S are well separated then S can be isotoped out of M''. After the isotopy, $\partial M''$ is an incompressible surface in the complement of S. Thus there is an incompressible torus, isotopic to ∂M in the complement of S. We showed that no such surface exists, so Σ and S are not well separated.

Now suppose $T' \cup T$ is incompressible. The arguments of Theorem 8 apply to this case as well, although considerably simplified by the fact that $T' \cup T$ is incompressible instead of weakly incompressible. The details of this case are left to the reader. □

Proof of Corollary 2. By Lemma 7, there is a knot K with unknotting tunnel τ such that for the induced Heegaard splitting Σ, $d(\Sigma) > 2n+2$. As noted in [4], every unknotting tunnel for a torus knot has distance at most 2, so K is not $(1,0)$. Thus by Theorem 1, K is not $(1,n)$. □

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